ISI Type-B Mock Test

There are 3 types of questions. Easy, Medium and Hard.
Easy = 4 points. Medium = 8 points. Hard = 12 points.
Answer as much as you can. Best of Luck!

1. (Easy points) Prove that;
    1. 2^n can be written as sum of two consecutive odd integers.
    2. 3^n can be written as sum of three consecutive integers.
    3. **Home Problem:** Can you generalize?
2. (Medium points) Let $S(x)$ be the sum of digits of the number x, in its decimal representation.
    1. Show that, $\dfrac{S(x)}{S(2x)} \leq 5$. Find a number x for which this bound is achieved.
    2. Prove that, $\dfrac{S(x)}{S(3x)}$ is unbounded. (Some credits will be given if you can write the precise mathematical statement that we need to show).
3. (Hard points) Show that the sum
$$S(m,n) = \sum_{k=1}^{n} \frac{1}{(m+k)}$$
is not an integer for any given positive integers m and n.
4. (Medium points) Inside a circle of radius 1 unit, there is a circle of diameter 1 unit. On the perimeter of this smaller circle, there is a fixed point P. Then find the locus of P, as the smaller circle **rolls** being tangent to the larger circle from inside. This is called **Copernicus Theorem**.
Home Problem: What is the locus if the smaller circle rolls being tangent to the inner circle from outside.
5. (Medium points) Let L_1 and L_2 be two lines. Then, find the set of points for which the sum of distances from the point to the lines is given to be equal to 1 unit.
Hint: Please recheck. Did you consider all the cases?
6. (Easy points) Show that;
$$\binom{n}{k+1}\binom{n}{k-1} \leq \binom{n}{k}^2$$
Home Problem: Try to show that by counting argument, by considering number of possible paths on a grid.
7. (Hard points) We define the valid coloring of a graph of V vertices and E edges to color the vertices such that no two vertices that share an edge are colored the same. Consider the following graph:
    1. The vertices are the integers 2, 3, 4, 5, 6, …., 9,998, 9,999, and 10,000.
    2. Two vertices are connected by an edge if they share a common factor. So there isn't an edge between 2 and 3, but there is between 2 and 6.

    Prove that the a valid colouring of this graph requires at least 13 colours.